length is in the order of several centimeters. The introduction of two-stream functions makes it possible to include the satellite speed in the self-consistent equations. This analysis determines the potential field around a moving satellite, which is shown not to have spherical symmetry. Furthermore, the determined potential field is significantly associated with the calculation of Coulomb drag, as indicted in Eq. (15). Two-stream instability predicts a long wake trail behind the satellite and high frequency of oscillation which agrees with the fact. By using the actual data of Explorer VIII[9] and the data of the upper atmosphere[3] (satellite speed $= 7.4 \times 10^3$ m/sec), the mean thermal speed of electrons $= 2.06 \times 10^5$ m/sec at the altitude of 1000 km, the mean thermal speed of ions $= 1.20 \times 10^3$ m/sec, and the temperature $= 1100$ °K; Eq. (13) then gives as tagnation potential of -0.156 v as compared with the actual measurement of -0.15 v. Of course, the agreement of the surface potential with the experimental one is not a test of the potential distribution, but only of Eq. (13) and, to a lesser degree, of the assumption concerning the surface interaction. However, the interpretation[9] of Explorer VIII data supports qualitatively the nonspherical potential field.

References

[1] Massey, H. S. W. and Burhop, E. H. S., *Electronic and Ionic Impact Phenomena* (Oxford University Press, London, 1952), pp. 555–565.

[2] Bird, G. A., "The flow about a moving body in the upper ionosphere," J. Aerospace Sci. **29**, 808–814 (1960).

[3] Brundin, C. L., "Effects of charged particles on the motion of an earth satellite," AIAA J. **1**, 2529–2538 (1963).

[4] Chapman, S. and Cowling, T. G., *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, London, 1960), pp. 69–88.

[5] Rand, S., "Wake of a satellite traversing the ionsphere," Phys. Fluids **3**, 265–273 (1960).

[6] Jackson, J. D., "Longitudinal plasma oscillations," J. Nucl. Energy: Pt. C. **1**, 171 (1960).

[7] Jastrow, R. and Pearse, C. A., "Atmospheric drag on the satellite," J. Geophys. Res. **62**, 413–423 (1957).

[8] Jen, N. C., "Plasma stress tensor on a wall," *Magnetohydrodynamics*, edited by N. Mather and G. Sutton (Gordon and Breach Scientific Publications, New York, 1962), pp. 551–558.

[9] Bordeau, R. E., Donley, J. L., Serbu, G. P., and Whipple, E. C., Jr., "Measurement of sheath currents and equilibrium potential on the Explorer VIII satellite," J. Astronaut Sci. **8**, 65–73 (1961).

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Optimum Estimation of Satellite Trajectories Including Random Fluctuations in Drag

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This paper considers the problem of trajectory estimation for a low-altitude satellite when random fluctuations in atmospheric density cause noticeable changes in the motion. The standard deterministic model for satellite motion is replaced by a stochastic model; optimum estimation theory is used to obtain the best fit to the data and the model. To test the efficacy of this approach to the problem, a particular stochastic drag model and the corresponding estimation procedure have been incorporated into a computer program called "orbit fit," which is similar to one used for actual flights. The effect of drag is calculated from a standard variation of parameters with an exponential atmosphere, whereas the stochastic part of the model is based upon a first-order stationary Gauss-Markov stochastic process. Both radar data and Baker-Nunn optical data from low-altitude satellite 1960 Omicron have been run on the program, and the results are shown to be a significant improvement over those using a deterministic model, i.e., the in-track residuals are reduced from thousands of feet to hundreds of feet. In addition, the consequence of changing the correlation and variance of the stochastic model is examined. The correlation had less effect over the rms residuals than the variance. In particular, for too small a variance, the residuals reduced to those of the deterministic model, whereas for too large a variance, the residuals resulted in unstable oscillations in the estimation procedure.

I. Introduction

THE problem of estimating the trajectory of a satellite can be stated as follows: Given a series of measurements of the position of a satellite, calculate the set of parameters which specifies the path of the satellite. When air drag can be neglected, the motion of the satellite is completely determined by six parameters. These six might be the position and velocity at some initial time, or a set of six orbit elements at the time of the ascending node. From the six parameters, the current and future motion can be determined by numerical integration of the acceleration due to the earth's gravitational field or from a closed-form approximation to the integrated motion. In general, when drag is included, the assumption is usually made that its effect can also be represented as a deterministic phenomenon (for instance, as a polynomial with undetermined coefficients). This treatment may be adequate as long as drag is not a significant factor in the satellite motion. However, for many satellites, the deterministic model cannot represent the motion correctly. Fluctuations in drag (sometimes correlated with solar activity) can cause substantial

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changes in the satellite orbit, particularly when there are gaps between measurements.

The purpose of this paper is to present a procedure for trajectory estimation that uses a stochastic model to represent the drag. The procedure makes use of some recent work in optimum estimation theory; this work has given a solution to the problem of prediction and smoothing in a form that can be easily mechanized on a digital computer.[1–5] The particular stochastic model developed in this paper is based on the assumption that the statistical properties of the atmosphere can be adequately represented by a first-order stationary Gauss-Markov process. Therefore, the satellite motion can be determined by eight parameters that are a set of six mean orbit elements, a seventh parameter that is related to the constant average value of the drag, and an eighth parameter that is related to the instantaneous value generated by the stochastic process.

The drag model and the estimation procedure have been incorporated into a computer program called orbit fit, which is similar to one used for actual flights. Both radar data and Baker-Nunn optical data taken during the first three days of satellite 1960 Omicron have been run on the program. (This satellite was chosen because, during the time of its launch, there were large fluctuations in air density due to solar activity.) The in-track residuals of the smoothed trajectory based upon radar measurements alone were reduced by an order of magnitude over the residuals using a constant drag model, i.e., from thousands of feet to hundreds of feet. Additional data from Baker-Nunn optical sightings confirm this improvement. These results are summarized in Sec. V.

Stochastic models of drag have been used before to analyze errors in orbit prediction,[6, 7] but, it is believed, that this paper represents the first use of optimum estimation theory in combination with the stochastic model to improve the accuracy of the resulting trajectory. If the fluctuations in drag can be adequately represented by the stochastic model, the resulting trajectory will be the optimum estimate based on the available data. This estimate has the following advantages: 1) large amounts of data can be combined with relatively small residuals; 2) the estimated trajectory can respond to short-term fluctuations in drag (of several hours); and 3) the statistical characteristics of the fluctuations are summarized by two quantities that have a physical meaning: the variance (how big) and the correlation (how long). The stochastic model for the drag fluctuations may also prove valuable in analyzing the variations in the density of the upper atmosphere. Currently, quite ingenious techniques are used to obtain longer term fluctuations in density,[8] but it may be necessary to modify these to analyze short-term fluctuations. Finally, as deterministic sources of error are reduced (such as uncertainties in station location and lack of knowledge about the earth's gravitational potential), it is felt that precision orbit determination will require a more careful analysis of stochastic errors such as those due to drag.

II. Estimation Procedure

This section presents the recent results in optimum estimation theory which have been applied to the orbit determination problem. Since the pioneer work of Wiener[9] on the problem of linear smoothing, filtering, and prediction, many papers have appeared giving different solutions to the problem. A summary of these solutions can be found in a paper by Parzen,[10] who gives a general treatment from the point of view of reproducing kernel Hilbert space. The filtering and prediction solution presented here is the one derived by Kalman,[1] which is probably the most widely used solution in practice. The primary advantage of Kalman's solution is that the equations that specify the optimum filter are in the form of difference equations; hence they can be mechanized easily on a present day digital computer. However, Kalman does not consider the important problem of smoothing. (The filtering and prediction solution allows one to estimate the current and future position of the satellite, whereas the smoothing solution permits one to estimate the past history of the satellite motion.) There are at least three different solutions to the smoothing problem,[3–5] and the results presented here were derived by Rauch, Tung, and Strebel.[5] This smoothing solution is also in the form of difference equations, and it uses the set of solutions to the filtering problem in the calculation of the smoothed estimate.

In formal terms, the problem can be stated as follows: given a linear dynamic system†

$$\begin{aligned} x(k+1) &= \Phi(k+1,k)x(k) + w(k) \\ y(k) &= M(k)x(k) + v(k) \end{aligned} \tag{1}$$

where $x(k)$ is the $(n \times 1)$ state vector, $y(k)$ is the $(m \times 1)$ output or observation vector (with $m \leq n$), $\Phi(k+1,k)$ is the $(n \times n)$ transition matrix, $w(k)$ and $v(k)$ are $(n \times 1)$ and $(m \times 1)$ vector-valued independent normal random variables with zero mean and covariances given by

$$\begin{aligned} \mathrm{cov}[w(j), w(k)] &= Q(k)\delta_{jk} \\ \mathrm{cov}[v(j), v(k)] &= R(k)\delta_{jk} \\ \mathrm{cov}[w(j), v(k)] &= 0 \end{aligned} \tag{2}$$

where δ_{jk} is the Kronecker delta and we assume that $R(k)$ is positive definite. The a priori knowledge about the initial condition $x(0)$ shows it to be a normal random variable with mean $\bar{x}(0)$ and covariance $\bar{P}(0)$. The problem is to find the best estimate of $x(k)$ given the series of observations $y(1), y(2), \ldots y(N)$. It is commonly called the problem of prediction if $k > N$, if filtering if $k = N$, and of smoothing if $k < N$.

For the satellite problem considered in this paper, the state x is an eight vector, with the first six components a set of mean orbit elements, and the last two related to the drag. The random variable w represents the effect of the random fluctuations in drag, whereas the random variable v is the noise on the measurements. The measurements $y(k)$ are the deviation of the actual observations from some set of nominal values, and the output matrix $M(k)$ is the set of partial derivatives of the observations with respect to the orbit elements. The resulting estimate is optimum in the sense that it is the conditional mean of the state vector and, hence, it minimizes a large class of loss functions including the mean-squared-error. For the smoothing solution, the optimum estimate is the one which minimizes the loss function J‡:

$$J = \sum_{k=1}^{N} ||y(k) - M(k)x(k)||^2_{R(k)^{-1}} + \sum_{k=1}^{N} ||x(k) - \Phi(k,k-1)x(k-1)||^2_{Q(k)^{-1}} + ||x(0) - \bar{x}(0)||^2_{\bar{P}(0)^{-1}} \quad 3))$$

When there is no a priori knowledge about initial conditions, the last term in Eq. (3) is zero. For the case where the random variable w is identically zero and where there is no a priori knowledge about the initial conditions, the optimum estimate is also the maximum likelihood estimate of the initial value of the state (which gives the parameters that determine the motion). For the more general case, one might say that the optimum estimate is the modified maximum likelihood estimate because the conditional mean of a normal random variable is the "most likely" point.

The optimum estimate of $x(k)$ given the observations $y(1), y(2), \ldots y(N)$ will be denoted by $\hat{x}(k/N)$ and the covariance of the estimate will be denoted by $P(k/N)$:

$$P(k/N) = \mathrm{cov}[x(k) - \hat{x}(k/N)] \tag{4}$$

† When the system is described by nonlinear equations, the linear system is obtained from equations governing small deviations from a reference trajectory.

‡ $||a||^2_R = a^T R a$.

The solution to the filtering and prediction problem as derived by Kalman is

$$\hat{x}(k+1/k) = \Phi(k+1,k)\hat{x}(k/k)$$
$$\hat{x}(k/k) = \hat{x}(k/k-1) + B(k)[y(k) - M(k)\hat{x}(k/k-1)] \tag{5}$$

where $B(k)$ is an $(n \times m)$ weighting matrix, which is calculated at each point from the covariance matrix at that point:

$$\left.\begin{aligned} B(k) &= P(k/k-1)M^T(k) \times [M(k)P(k/k-1)M^T(k) + R(k)]^{-1} \\ P(k+1/k) &= \Phi(k+1,k)P(k/k) \times \Phi^T(k+1,k) + Q(k) \\ P(k/k) &= [I - B(k)M(k)]P(k/k-1) \end{aligned}\right\} \tag{6}$$

A block diagram of the filtering solution is presented in Fig. 1. Notice how that part of the diagram enclosed by the dashed lines duplicates the model of the system in Eq. (1). The filtered estimate at each point is calculated as a linear combination of the previous estimate and the current measurement. Therefore, starting with the initial conditions $\bar{x}(0)$ and $\bar{P}(0)$, Eqs. (5) and (6) can be used recursively by processing the measurements in sequential order to find the filtered estimate at each point and the associated covariance.

The solution to the smoothing problem is

$$\hat{x}(k/N) = \hat{x}(k/k) + C(k)[\hat{x}(k+1/N) - \hat{x}(k+1/k)]$$
$$C(k) = P(k/k)\Phi^T(k+1,k)P^{-1}(k+1/k) \tag{7}$$

The $(n \times m)$ weighting matrix $C(k)$ is calculated at each point from the covariance of the filtering estimate. Furthermore, if it is desired, the covariance of the smoothed estimate can be calculated by

$$P(k/N) = P(k/k) - C(k)[P(k+1/k) - P(k+1/N)]C^T(k) \tag{8}$$

A block diagram of the smoothing solution is presented in Fig. 2. The smoothing solution is in the form of a backward difference equation that starts with the filtered estimate at the last point (which is also the smoothed estimate at that point). It then calculates backward point by point from Eq. (7) determining the smoothed estimate as a linear combination of the filtered estimate at that point and the smoothed estimate at the previous point. In summary, the estimation procedure followed in this paper is: 1) use the filtering solution derived by Kalman (a forward difference equation) to determine the filtered estimate at each point, starting at the first point and going to the last, and 2) use the smoothing solution (a backward difference equation) to determine the smoothed estimate at each point, starting at the last point and going back to the first.

III. Drag Model

The position and velocity of the satellite at any time is determined from a closed-form approximation to the integrated motion, which is based on the equations of variation for six mean orbit elements. The elements would be constant if there were no drag. The change of these six elements with drag is calculated by variation of parameters assuming an atmosphere which varies exponentially with height h. Any additional changes in drag (such as fluctuations due to solar activity or insolation of perigee) can be superimposed on top of this model. The instantaneous acceleration due to drag is assumed to be opposed to the direction of motion and proportional to the air density ρ and the velocity squared V^2:

$$D = -\rho V^2/2C \qquad \rho = \rho_p e^{-k(h-h_p)} \tag{9}$$

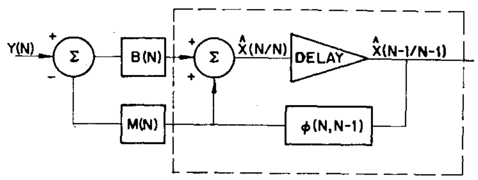

Fig. 1 Block diagram of the filtering solution.

where C is the hypersonic ballistic coefficient, ρ_p and h_p are the density and height at perigee, and $1/k$ is the scale height. It is assumed that the change in the mean elements is approximated by the change in the corresponding set of osculating elements. The variation of parameter equations for the change with time in the instantaneous osculating elements are[11]

$$\left.\begin{aligned} (1/a)(da/dt) &= -(\rho v/C)(1 + 2e\cos\theta^* + e^2)/(1-e^2) \\ de/dt &= -(\rho v/C)(\cos\theta^* + e) \\ e(d\omega/dt) &= -(\rho v/C)\sin\theta^* \\ di/dt &= 0 \\ d\Omega/dt &= 0 \end{aligned}\right\} \tag{10}$$

where a, e, ω, i, and Ω are the semimajor axis, eccentricity, argument of perigee, inclination, and right ascension of the ascending node, and θ^* is the difference between the true anomaly θ and the argument of perigee. The variation of the time of the ascending node T_Ω can be obtained from Eq. (10) and the time equation

$$t - T_\Omega = (a^3/\mu)^{1/2}(E - e\sin E - E_\Omega + e\sin E_\Omega)$$
$$E = \theta^* - 2\arctan\left[\frac{e\sin\theta^*}{1 + (1-e^2)^{1/2} + e\cos\theta^*}\right] \tag{11}$$

where E is the eccentric anomaly at time t and E_Ω is the eccentric anomaly at the time of the ascending node T_Ω, when θ is equal to zero. The resulting equation for the variation of T_Ω shows the change in period caused by drag:

$$\left(\frac{\mu}{a^3}\right)^{1/2}\frac{dT_\Omega}{dt} = -\frac{3}{2}\left(\frac{\mu}{a^3}\right)^{1/2}(t - T_\Omega)\frac{1}{a}\frac{da}{dt} + (1-e^2)^{1/2}\left[\frac{\sin\theta^*(2 + e\cos\theta^*)}{(1+e\cos\theta^*)^2} + \frac{\sin(2+e\cos\omega)}{(1+e\cos\omega)^2}\right]\frac{de}{dt} + (1-e^2)^{3/2}\left[\frac{2\cos\omega - 2\cos\theta^* + e(\cos^2\omega - \cos^2\theta^*)}{(1+e\cos\theta^*)^2(1+e\cos\omega)^2}\right]e\frac{d\omega}{dt} \tag{12}$$

The state vector x, which determines the motion, has eight components. The first six represent a normalized set of six mean orbit elements, the seventh component represents the constant average effect of drag and is proportional to the mean rate of decay of the period, and the eighth represents the stochastic effect of drag. Specifically, the first six components are given by

$$\left.\begin{aligned} x_1 &= (a/a_N) - 1 & x_4 &= (\mu/a_N^3)^{1/2}(T_\Omega - T_{\Omega N}) \\ x_2 &= e\cos\omega & x_5 &= i \\ x_3 &= e\sin\omega & x_6 &= \Omega \end{aligned}\right\} \tag{13}$$

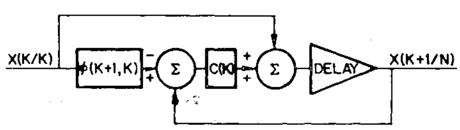

Fig. 2 Block diagram of the smoothing solution.

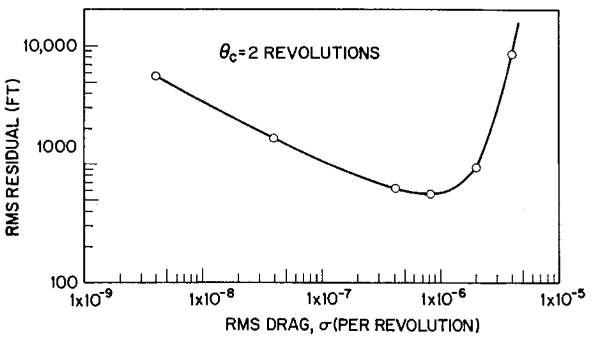

Fig. 3 Root-mean-square in-track residuals vs root-mean square drag.

where the orbital elements in Eq. (13) represent mean rather than osculating orbit elements, and a_N and $T_{\Omega N}$ are nominal values (which are very close to the true values) of the mean semi-major axis and the mean time of nodal crossing. Under the assumption that the variation for the osculating orbit elements in Eq. (10) can be used for the mean orbit elements in Eq. (13) and that eccentricity e is small, the change in the first six components of the state is

$$dx_1/d\theta = -(x_7 + x_8)g(\theta)$$
$$dx_2/d\theta = -(x_7 + x_8)g(\theta) \cos\theta$$
$$dx_3/d\theta = -(x_7 + x_8)g(\theta) \sin\theta$$
$$dx_4/d\theta = -(x_7 + x_8)g(\theta)[3\theta/2 - 2 \sin\theta]$$
$$dx_5/d\theta = 0 \qquad dx_6/d\theta = 0$$

$$\left.\begin{aligned} [g(\theta) &= e^{kae\cos\theta^*}/[I_0(kae)] \\ I_0(kae) &= \frac{1}{2\pi}\int_0^{2\pi} e^{kae\cos\theta^*}\, d\theta \end{aligned}\right\} \qquad (14)$$

where I_0 is the modified Bessel function of order zero. A more complete derivation of Eq. (14) is given in the Appendix. Notice that the average of $g(\theta)$ over one revolution is unity. The seventh component of the state x_7 is assumed to be constant, although it can be time varying if there is additional information concerning the atmospheric density (such as insolution of perigee). The eighth component of the state x_8 is a stochastic variable, which is generated by a first-order stationary Gauss-Markov process so that

$$\begin{aligned} E[x_8] &= 0 \\ E[x_8(\tau + \theta)x_8(\tau)] &= \sigma^2 e^{-\theta/\theta_c} \quad \text{for } \theta \geqslant 0 \end{aligned} \qquad (15)$$

where $E[A]$ is the expected value of A and where σ^2 and θ_c will be called the variance and the correlation of the stochastic model for drag, respectively. The quantity θ_c is actually the distance or time that it takes the autocorrelation function to decrease to $1/e$ times its initial value, but in this paper it will be called the correlation.

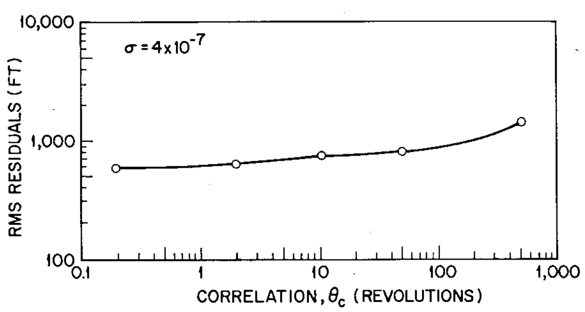

Fig. 4 Root-mean-square in-track residuals vs correlation.

The transition matrix $\Phi(k + 1,k)$ and the covariance $Q(k)$ and $R(k)$, along with the measurements $y(k)$ and the partial derivatives $M(k)$ completely specify the estimation problem. The transition matrix $\Phi(k + 1, k)$ can be calculated by integrating Eqs. (14) and (15) directly. For a first-order stochastic process, the covariance $Q(k)$ can be calculated easily. For instance, after an angle θ, the variance of x_1 given by the q_{11} component of the covariance Q is

$$q_{11} = \sigma^2 \int_0^\theta \int_0^\theta g(\zeta)g(\eta) \left\{\exp\left[-\frac{|\zeta - \eta|}{\theta_c}\right] - \exp\left[-\frac{(\zeta + \eta)}{\theta_c}\right]\right\} d\zeta d\eta \qquad (16)$$

The covariance $R(k)$, the measurements $y(k)$, and the partial derivatives $M(k)$ are the same as the deterministic model.

If it is necessary, the stochastic model can be expanded to a higher-order process, or made nonstationary. For instance, it can be expanded to a second-order process including long term as well as short term random fluctuations (as is done in reliability calculations), or it can be made time-varying to respond to daily effects. In the computer program, the stationary first-order model has been used to avoid additional complications.

IV. Computer Program

The random drag program is the modification of a program called orbit fit developed by J. V. Breakwell. Orbit fit is a research version of a program used to support actual flights. It is used in postflight analysis and the input data can be Baker-Nunn optical observations as well as condensed radar data. The condensed radar data give the position and velocity resulting from a least-squares fit of the individual range and angle measurements from the radar station. The optical observations are input as individual measurements.

The program uses a closed-form solution to the equations of motion, which includes both zonal and tesseral harmonics of the earth's gravitational potentials. The closed-form solution includes terms of order eccentricity times oblateness parameter ($e{\cdot}J_2$) but neglects terms of order eccentricity times the higher harmonics. This representation is quite accurate for low eccentricities. For the results presented in Sec. V, the inputs were recent values of the even zonal harmonics up to J^{12}, the odd zonal harmonics up to J^7, and the tesseral harmonics $J_2{}^2$, $J_3{}^3$, and $J_4{}^4$. The orbit elements in the closed-form solution are not osculating Keplerian elements, but they are certain mean orbital elements.[12] The effect of drag is included separately by treating the mean orbital elements as if they were instantaneous elements and by using variation of parameters as outlined in Sec. III.

The estimation procedure is a sequential one. At each measurement point the estimate of the eight parameters and the covariance of the estimate (the inverse of the "information matrix") are updated. The estimation procedure for the stochastic model differs from that for the deterministic model in that the effect of random fluctuations in drag is included by estimating an eighth component of the state, and the covariance matrix at each point is updated by including the loss in information due to the random fluctuations in the elements. The calculation of the covariance of the random fluctuations is done by combining numerical integration with a closed-form expression so that it takes almost no additional time.

The measurement at that point is used to make a differential correction of the estimate, and the covariance is modified to include the new information. The quantity to be minimized is a nonlinear function of the parameters, and the differential correction is based upon partial derivatives, so the correction procedure is repeated two more times at each measurement point (until there is no additional correction). After the last measurement, the program goes through a "backwards" estimation procedure in which the forward estimate of the eight parameters at each point is replaced by a smoothed estimate based on all the data. The program is primarily intended for postflight analysis, including optical observations, but the forward estimation procedure can be used to simulate an operational program supporting a real flight where the data is processed sequentially and the future motion must be predicted.

The entire estimation procedure requires about 20 sec on the IBM 7090 for each data station so that a total of nineteen stations uses about 6 min. The 20 sec includes the time for repeating the differential correction twice in the forward procedure as well as for making one correction in the background estimation or smoothing procedure.

V. Numerical Results

The random drag program has been tested by using data taken during the first three days of satellite 1960 Omicron.[13] The satellite was launched on November 12, 1960, with a period of 96.4 min, and was up for only 47 days. The apogee height, perigee height, and inclination were 614 statute miles, 113 statute miles, and 81.9°, respectively. Over the first three days the average constant drag was 24×10^{-6} fractional decay of the semimajor axis per revolution, which corresponds to a period decay of about 0.2 sec/rev. This satellite was chosen because fluctuations in atmospheric density over the first two days caused a drag change of over 50% from the first day to the second, and because there were a large number of highly accurate Baker-Nunn optical sightings of the satellite. The total data consist of 13 radar passes and six optical sets of optical sightings, spaced so that there is a gap of about 9 hr between the first and the second day when there were no measurements. (There was also a seventh set of optical sightings, which was discarded because it was completely inconsistent, by scores of miles, with all the rest of the data.)

The program was run with both optical and radar data, but in the results presented here the radar data were used to determine the orbit, whereas the optical data were used as a check. This approach was particularly appropriate because the optical data were highy accurate (perhaps to less than one hundred feet) and also because the radar data were concentrated mainly in the northern hemisphere, whereas the optical data were mainly in the southern hemisphere. With a given set of data, the quantities that can be varied are the initial conditions [the initial estimate $\bar{x}(0)$ and its covariance $\bar{P}(0)$], the rms drag of the stochastic model (σ), and the correlation of the stochastic model (θ_c). For all the results presented here the initial conditions were held fixed while only the rms drag and correlation of the model were changed.

Since the most important effect of drag is to change the time equation, it is the in-track residuals that should be examined.§ The weighted rms in-track residuals for radar data alone are plotted in Fig. 3 as a function of rms drag with the correlation held fixed at two revolutions (about 3 hr). For very low rms drag, the stochastic model reduces to the deterministic model and the weighted rms residuals approach the value obtained using the deterministic model (6800 ft). Notice that the rms residuals for the stochastic model reach a minimum at about one-tenth of the value obtained using the deterministic model. For very high rms drag, the forward estimaton procedure tended to be marginally stable. The high value of stochastic drag was used to overcorrect the estimate first one way then the other. The rms in-track residuals for radar data alone are plotted again in Fig. 4 as a function of the correlation with the rms drag held fixed at 4×10^{-7}.

§ Because the radar range measurements are so much more accurate than the angle measurements, the in-track residuals in position are determined by comparing the in-track position at the time when the range reaches a minimum (the point of closest approach).

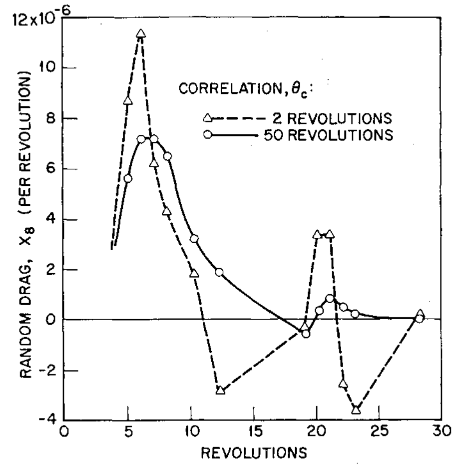

Fig. 5 Stochastic drag vs revolutions.

The results show that the rms residuals were not very sensitive to the correlation. Once again, as the correlation gets very large, the stochastic model will approach the deterministic model. The estimate of the stochastic drag is plotted in Fig. 5 as a function of revolutions for two values of the correlation with σ held fixed at 4×10^{-7}. Notice how the larger value of the correlation tends to mask the rapid fluctuations in the drag.

A point-by-point comparison of the in-track residuals at each radar pass is presented in Table 1 for the deterministic model with rms drag zero and the stochastic model with the rms drag 4×10^{-7} and the correlation 10 rev. Comparing the two sets of results shows that the stochastic treatment of drag has reduced the residuals from the deterministic treatment using constant drag by an order of magnitude from thousands of feet to hundreds of feet. In order to check the radar results, the optical data were included with zero weight, i.e., the residuals of the optical sightings were calculated, but no differential corrections were made. The residuals at three of the optical stations were less than 200 ft and at the fourth station, following the gap of 9 hr with no measurements, the residuals were 1500 ft. The prediction of the optical measurements after 3 rev and 6 rev following the end of the radar data gave residuals of 28,000 and 70,000 ft, respectively, in-

Table 1 Comparison of in-track residuals, 10^3 ft

	Data[a]	Revolution[b]	Deterministic[c]	Stochastic[d]
1	R	0.4	44.6	2.7
2	O	4.0	...	0.1
3	R	5.1	5.7	0.9
4	R	6.1	0.3	−0.3
5	R	7.1	−1.6	0.5
6	R	8.1	−3.0	0.4
7	R	11.4	−4.4	0.2
8	R	13.4	2.1	0.4
9	O	19.0	...	1.5
10	R	19.1	2.6	0.7
11	O	20.1	...	0.2
12	R	20.1	1.6	0.4
13	O	21.1	...	0.1
14	R	21.1	0.6	−0.1
15	R	22.1	2.6	2.1
16	R	23.1	0.6	0.3
17	R	28.4	4.6	0.1
18	O	31.1	...	28.0
19	O	34.1	...	70.0

[a] Optical = O and radar = R.
[b] Since time of first ascending node.
[c] Radar data with constant drag model.
[d] Radar data with zero weight on optical (for $\sigma 0.4 \times 10^{-6}$ and θc, 10 rev).

dicating that there still might be some stochastic fluctuations on the third day. Also, a capsule was separated from the satellite on the 31 orbit, and this may have influenced the drag characteristics.[13]

VI. Conclusions

For satellite 1960 Omicron, the stochastic model for drag and the corresponding estimation procedure have shown a striking improvement over the deterministic model for drag. Even when the variance and correlation of the stochastic model are orders of magnitude different than the best value, there are still significant improvements over the deterministic model. For other satellites, the difference in the two approaches may not be so marked, but for any low-altitude satellite, there should be some significant advantage to using the stochastic model for drag when high-accuracy trajectory determination is involved.

Appendix

In this appendix, the steps leading up to Eq. (13) will be developed for the decay of the semimajor axis. The remainder of the expressions in Eq. (13) follow directly from analogy with Eq. (10). In terms of osculating elements, the expression for decay of the semimajor axis from Eq. (10) is

$$(1/a)(da/dt) = -(\rho_p v/c)e^{-k(h-h_p)} \quad \text{(A1)}$$

From the expression for the radius and its derivative,

$$\left.\begin{aligned} r &= [a(1-e^2)]/(1+e\cos\theta^*) \\ h - h_p &= r - a(1-e) = \frac{a(1-e)e\,(1-\cos^*\theta)}{1+e\cos\theta^*} \\ v &= \left[\left(r\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2\right]^{1/2} = \\ &\frac{rd\theta}{dt}\left(1 + \frac{e^2\sin^2\theta^*}{(1+e\cos\theta^*)^2}\right)^{1/2} \end{aligned}\right\} \quad \text{(A2)}$$

For small eccentricity, the quantity e can be neglected in comparison to one, so that

$$\begin{aligned} h - h_p &= ae(1-\cos\theta^*) \\ v &= a(d\theta/dt) \end{aligned} \quad \text{(A3)}$$

Substituting Eq. (A3) into Eq. (A1) yields

$$(1/a)(da/d\theta) = -(\rho_p a/c)e^{-kae}e^{+kae\cos\theta^*} \quad \text{(A4)}$$

The zero-order modified Bessel function can be written as

$$I_0(kae) = \frac{1}{2\pi}\int_0^{2\pi} e^{kae\cos\theta^*}\,d\theta \quad \text{(A5)}$$

If the only random variable is the atmospheric density at perigee, the expected value of the constant average decay of the semimajor axis (which is the component of the state x_7) can be written as

$$\begin{aligned} x_7 &= -E\left[\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{a}\frac{da}{d\theta}\,d\theta\right] \\ &= E[\rho_p](a/c)e^{-kae}I_0(kae) \end{aligned} \quad \text{(A6)}$$

Since the instantaneous decay is defined by Eq. (13) as

$$\frac{1}{a}\frac{da}{d\theta} = -(x_7 + x_8)\frac{e^{+kae\cos\theta^*}}{I_0(kae)} \quad \text{(A7)}$$

substituting Eqs. (A6) and (A7) into Eq. (A4) and solving for x_7 yields

$$x_7 = (\rho_p - E[\rho_p])(a/c)I_0(kae)e^{-kae} \quad \text{(A8)}$$

References

[1] Kalman, R. E., "A new approach to linear filtering and prediction problems," J. Basic Eng. **82,** 35–45 (March 1960).

[2] Kalman, R. E. and Bucy, R. S., "New results in linear filtering and prediction theory," J. Basic Eng. **83,** 95–108 (March 1961).

[3] Bryson, A. E. and Frazier, M., "Smoothing for linear and nonlinear dynamic systems," *Proceedings of the Optimum System Conference*, Wright-Patterson Air Force Base, Ohio, Aeronautical Systems Division, TDR-63-119, pp. 353–364 (September 1962).

[4] Rauch, H. E., "Solutions to the linear smoothing problem," Inst. Elec. Electron. Engrs. Trans. Auto. Control **AC8,** 371–372 (October 1963).

[5] Rauch, H. E., Tung, F., and Striebel, C. T., "On the maximum likelihood estimates for linear dynamic systems," Joint AIAA-IMS-SIAM-ONR Symposium on Control and System Optimization, Monterey, Calif. (January 27–29, 1964); also Lockheed Missiles and Space Co., Palo Alto, Calif., TR 6-90-63-62 (June 1963).

[6] Breakwell, J. V., "Accuracy of orbit determination," Lockheed Missiles and Space Co., Palo Alto, Calif., Rept. LMSD/-704013 (October 1960).

[7] Moe, K., "Stochastic models of the errors in orbital prediction for artificial earth satellites," ARS **J. 32,** 1726–1728 (1962).

[8] Rossoni, J. P., Sconzo, P., Greenfield, R. J., and Champion, K. S., "Atmospheric density determination from satellite observations," AIAA Preprint 64-31 (January 1964).

[9] Wiener, N., *The Extrapolation, Interpolation, and Smoothing of Stationary Time Series* (John Wiley and Sons, New York, 1949).

[10] Parzen, E., "An approach to time series analysis," Ann. Math. Statist. **32,** 951–989 (1961).

[11] Danby, J. M., *Fundamentals of Celestial Mechanics* (The Macmillan Co., New York, 1962).

[12] Petty, C. M. and Breakwell, J. V., "Satellite orbits about a planet with rotational symmetry," J. Franklin Inst. **270,** 259–282 (October 1960).

[13] Space Log **3,** 6–7 (December 1963).